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# $R-$Majorizing Quadratic Stochastic Operators: Examples on 2D Simplex

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## ABSTRACT

A vector majorization is a preorder of dispersion for vectors with the same length and same sum of components. The vector majorization can be viewed as a preorder of distance from a uniform vector. A preorder of distance from any fixed non-uniform vector of positive components, so-called $r-$majorization, is a generalization of usual vector majorization. In this paper, a new class of mappings so-called $r-$majorizing quadratic stochastic operators was introduced. The $r-$majorizing quadratic stochastic operator is a generalization of a quadratic doubly stochastic operator. Some relevant examples are provided. Moreover, the dynamics of some non-scrambling $r-$majorizing quadratic stochastic operators are studied.

**Keywords:** $R-$majorization, quadratic stochastic operators, scrambling matrix.

# 1. Introduction

The dynamics of nonlinear operators remains to be difficult and complex. The simplest nonlinear operator is a quadratic operator. The quadratic stochastic operator (in short QSO) has an incredible application in population genetics (Lyubich, 1992). The QSO describes a distribution of the next generation in the population system if the distribution of the current generation was given. The QSO is a primary source for investigations of evolution of population genetics. In Ganikhodjaev et al. (2013), a mathematical model of a transmission of human ABO blood groups was described as the QSO on 7-dimensional simplex and based on some numerical investigations of QSO, the future ABO blood group distribution of Malaysian people was predicted. In Ganikhodzhaev et al. (2011), it was given a long self-contained exposition of the recent achievements and open problems in the theory of QSO.

The main problem in the nonlinear operator theory is to study its behavior. This problem was not fully finished even in the class of QSO (Ganikhodzhaev et al., 2011), (Mukhamedov and Saburov, 2010), (Mukhamedov and Saburov, 2014), (Mukhamedov et al., 2013), (Saburov, 2013). In this paper, a new class of mappings so called $r-$majorizing QSO was introduced. The dynamics of any QSO on 1D simplex is more or less clear (Lyubich, 1992). However, there are many QSO on 2D simplex which remain to be investigated (Mukhamedov et al., 2013). Therefore, we are aiming to study the dynamics of $r-$majorizing QSO on 2D simplex.

A vector majorization is a preorder of dispersion for vectors with the same length and same sum of components. The vector majorization can be viewed as a preorder of distance from a uniform vector. A preorder of distance from any fixed non-uniform vector of positive components, so-called $r-$majorization, is a generalization of usual vector majorization. Several equivalent definitions of $r-$majorizations and related concepts are discussed in Joe (1990). Let us provide some necessary notions and notations related to $r-$majorizations.

Throughout this paper, we write vectors in the row forms.

Let $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$. We write $\mathbf{p} \geq 0$ (resp. $\mathbf{p} > 0$) whenever $p_i \geq 0$ (resp. $p_i > 0$) for all $i = \overline{1,3}$. Let $\|\mathbf{x}\|_1 = \sum\limits_{i=1}^{3} |x_i|$ be a norm of any $\mathbf{x} \in \mathbb{R}^3$. Let $\mathbb{S}^2 = \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \geq 0,\ \|\mathbf{x}\|_1 = 1\right\}$ be a standard simplex. An element of the simplex $\mathbb{S}^2$ is called *a stochastic vector.*

**Definition 1.1.** *Let* $\mathbf{x}, \mathbf{y}$ *and* $\mathbf{p} > 0$ *be stochastic vectors. We say that* $\mathbf{x}$ *is*

*$r-$majorized by* $\mathbf{y}$ *with respect to (w.r.t.)* $\mathbf{p}$ *(written* $\mathbf{x} \prec^r_{\mathbf{p}} \mathbf{y}$*) if one has that*

$$\sum_{i=1}^{3} |x_i - tp_i| \leq \sum_{i=1}^{3} |y_i - tp_i|, \quad \forall\ t \in \mathbb{R}. \tag{1}$$

The $r$ in $r-$majorization stands for *relative* or *ratio*.

Note that if $\mathbf{p} = \mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ then the $r-$majorization w.r.t. $\mathbf{p}$ is nothing but usual majorization (Joe, 1990),(Marshall et al., 2011). In this case, we shall use usual notation $\prec$ for the $r-$majorization.

A matrix is said to be *stochastic* (resp. *doubly stochastic*) if its rows (resp. its rows and columns) are stochastic vectors. We denote the set of all stochastic (resp. doubly stochastic) matrices by $\mathsf{SM}$ (resp. $\mathsf{DSM}$). Let us introduce the following set of stochastic matrices for a positive stochastic vector $\mathbf{p} > 0$

$$\mathsf{SM}[\mathbf{p}] = \{\mathbb{P} \in \mathsf{SM} : \mathbf{p}\mathbb{P} = \mathbf{p}\}. \tag{2}$$

The set $\mathsf{SM}[\mathbf{p}]$ of all stochastic matrices having a common fixed distribution $\mathbf{p} > 0$ is a convex compact subset of the set of all stochastic matrices $\mathsf{SM}$. It is worth of mentioning that if $\mathbf{p} = \mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ then $\mathsf{SM}[\mathbf{c}]$ is nothing but a set of all doubly stochastic matrices, i.e., $\mathsf{SM}[\mathbf{c}] = \mathsf{DSM}$.

The following result was proven in Joe (1990), Marshall et al. (2011).

**Theorem 1.1.** *(Joe, 1990, Marshall et al., 2011) The following are equivalent*

*(i)* *One has that* $\mathbf{x} \prec^r_{\mathbf{p}} \mathbf{y}$;

*(ii)* *There is a stochastic matrix* $\mathbb{P} \in \mathsf{SM}[\mathbf{p}]$ *such that* $\mathbf{x} = \mathbf{y}\mathbb{P}$;

*(iii)* *One has that*

$$\sum_{i=1}^{3} q_i \varphi\left(\frac{x_i}{p_i}\right) \leq \sum_{i=1}^{3} q_i \varphi\left(\frac{y_i}{p_i}\right)$$

*for all convex continuous functions* $\varphi$.

## 2. $R$-Majorizing Quadratic Stochastic Operators

Let $\mathcal{Q} = (q_{ijk})_{i,j,k=1}^3$ be a cubic stochastic matrix, i.e.,

$$\sum_{k=1}^{3} q_{ijk} = 1, \quad q_{ijk} = q_{jik}, \quad q_{ijk} \geq 0, \quad \forall i, j, k = \overline{1,3}.$$

We define a quadratic stochastic operator (in short QSO) $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$, $\mathcal{Q}(\mathbf{x}) = (\mathcal{Q}(\mathbf{x})_1, \mathcal{Q}(\mathbf{x})_2, \mathcal{Q}(\mathbf{x})_3)$ associated with a given cubic stochastic matrix $\mathcal{Q} = (q_{ijk})_{i,j,k=1}^3$ as follows

$$\begin{aligned}
(\mathcal{Q}(\mathbf{x}))_1 &= x_1^2 q_{111} + x_2^2 q_{221} + x_3^2 q_{331} + 2x_1x_2q_{121} + 2x_1x_3q_{131} + 2x_2x_3q_{231} \\
(\mathcal{Q}(\mathbf{x}))_2 &= x_1^2 q_{112} + x_2^2 q_{222} + x_3^2 q_{332} + 2x_1x_2q_{122} + 2x_1x_3q_{132} + 2x_2x_3q_{232} \\
(\mathcal{Q}(\mathbf{x}))_3 &= x_1^2 q_{113} + x_2^2 q_{223} + x_3^2 q_{333} + 2x_1x_2q_{123} + 2x_1x_3q_{133} + 2x_2x_3q_{233}
\end{aligned} \quad (3)$$

**Remark 2.1.** *Here, we are using the same notation for the cubic stochastic matrix and for the associated QSO in order to show some correlation.*

We define the following stochastic vectors and square stochastic matrices associated with the cubic stochastic matrix $\mathcal{Q} = (q_{ijk})_{i,j,k=1}^3$ as follows

$$\mathbf{q}_{ij\bullet} = (q_{ij1}, q_{ij2}, q_{ij3}), \qquad \forall\, i, j = \overline{1,3}, \quad (4)$$

$$\mathbb{Q}_i = (q_{ijk})_{j,k=1}^3, \qquad \forall\, i = \overline{1,3}, \quad (5)$$

$$\mathbb{Q}_{\mathbf{x}} = \mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3, \qquad \forall\, \mathbf{x} \in \mathbb{S}^2. \quad (6)$$

**Remark 2.2.** *It is worth of mentioning that $\mathbb{Q}_{\mathbf{e}_i} = \mathbb{Q}_i$ for any vertex $\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ of the simplex $\mathbb{S}^2$, where $\delta_{ij}$ is Kronecker's delta symbol.*

It is easy to check that the QSO has the following vector and matrix forms

$$\begin{aligned}
\mathcal{Q}(\mathbf{x}) &= x_1^2 \mathbf{q}_{11\bullet} + x_2^2 \mathbf{q}_{22\bullet} + x_3^2 \mathbf{q}_{33\bullet} + 2x_1x_2\mathbf{q}_{12\bullet} \\
&\quad + 2x_1x_3\mathbf{q}_{13\bullet} + 2x_2x_3\mathbf{q}_{23\bullet}, \quad (7)\\
\mathcal{Q}(\mathbf{x}) &= \mathbf{x}\mathbb{Q}_{\mathbf{x}} = x_1 \cdot \mathbf{x}\mathbb{Q}_1 + x_2 \cdot \mathbf{x}\mathbb{Q}_2 + x_3 \cdot \mathbf{x}\mathbb{Q}_3, \quad (8)
\end{aligned}$$

where $\mathbf{x} \in \mathbb{S}^2$ and $\mathbb{Q}_{\mathbf{x}} = \mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3$ is a square stochastic matrix.

We write the following notation for the matrix form (8) of QSO

$$\mathcal{Q} = \mathbb{Q}_1 \oplus \mathbb{Q}_2 \oplus \mathbb{Q}_3 \quad (9)$$

Since $\mathbf{q}_{ij\bullet} = \mathbf{q}_{ji\bullet}$, $\forall\ i,j = \overline{1,3}$ we have the following relation

$$\mathbb{Q}_{\mathbf{x}} = \mathbb{Q}_1 x_1 + \mathbb{Q}_2 x_2 + \mathbb{Q}_3 x_3 = \begin{pmatrix} \mathbf{x}\mathbb{Q}_1 \\ \mathbf{x}\mathbb{Q}_2 \\ \mathbf{x}\mathbb{Q}_3 \end{pmatrix} \tag{10}$$

**Definition 2.1.** *The QSO $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ given by* (3) *is said to be $r-$majorizing w.r.t. a stochastic vector $\mathbf{p} > 0$, if one has that $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{p}]$ for i.e., all square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ have a common stationary distribution $\mathbf{p}$.*

Let

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbf{q}_{11\bullet} \\ \mathbf{q}_{12\bullet} \\ \mathbf{q}_{13\bullet} \end{pmatrix}, \ \mathbb{Q}_2 = \begin{pmatrix} \mathbf{q}_{12\bullet} \\ \mathbf{q}_{22\bullet} \\ \mathbf{q}_{23\bullet} \end{pmatrix}, \ \mathbb{Q}_3 = \begin{pmatrix} \mathbf{q}_{13\bullet} \\ \mathbf{q}_{23\bullet} \\ \mathbf{q}_{33\bullet} \end{pmatrix}, \ \mathbb{E} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

**Proposition 2.1.** *The QSO $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ given by* (3) *is $r-$majorizing w.r.t. $\mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ if and only if all square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ are doubly stochastic and $\mathbb{Q}_1 + \mathbb{Q}_2 + \mathbb{Q}_3 = \mathbb{E}$.*

*Proof.* Only if part. Let $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ be the $r-$majorizing with respect to $\mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We then get that $\mathbf{c}\mathbb{Q}_1 = \mathbf{c}\mathbb{Q}_2 = \mathbf{c}\mathbb{Q}_3 = \mathbf{c}$. This means that all square stochastic matrices $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ are doubly stochastic. Moreover,

$$\mathbb{Q}_{\mathbf{c}} = \frac{1}{3}\mathbb{Q}_1 + \frac{1}{3}\mathbb{Q}_2 + \frac{1}{3}\mathbb{Q}_3 = \begin{pmatrix} \mathbf{c}\mathbb{Q}_1 \\ \mathbf{c}\mathbb{Q}_2 \\ \mathbf{c}\mathbb{Q}_3 \end{pmatrix} = \frac{1}{3}\mathbb{E}.$$

Therefore, we get that $\mathbb{Q}_1 + \mathbb{Q}_2 + \mathbb{Q}_3 = \mathbb{E}$.

If part. Let $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ be square doubly stochastic matrices and $\mathbb{Q}_1 + \mathbb{Q}_2 + \mathbb{Q}_3 = \mathbb{E}$. We then have that $\mathbf{c}\mathbb{Q}_1 = \mathbf{c}\mathbb{Q}_2 = \mathbf{c}\mathbb{Q}_3 = \mathbf{c}$, i.e., $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{c}]$. Hence, $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ is $r-$majorizing w.r.t. $\mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. ☐

**Remark 2.3.** *An $r-$majorizing QSO $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ w.r.t. $\mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is called a quadratic doubly stochastic operator (Ganikhodzhaev, 1992). The dynamics of this kind of operators was studied in Saburov and Saburov (2014a) and Saburov and Saburov (2014b).*

# 3. Examples

In this section, we provide some examples for an $r-$majorizing QSO on 2D simplex.

**Example 3.1.** *Let* $\mathbf{p} \in \mathbb{S}^2$ *and* $\mathbf{p} > 0$. *Without loss of generality, we may assume that* $0 < p_3 \leq p_2 \leq p_1 < 1$. *Let* $s = p_2 + p_3$ *and* $t = \frac{p_3}{p_2}$. *It is clear that* $0 < s, t \leq 1$. *Let us define the following set*

$$\mathbb{S}_{\mathbf{p}} = \big\{\mathbf{x} \in \mathbb{S}^2 : 0 < x_3 < st, \quad 0 \vee (s-t) < x_2 < s \\ 0 \vee [p_1(1+t) - t] < x_1 < 1 \wedge [p_1(1+t)]\big\},$$

*where* $a \vee b = \max\{a, b\}$ *and* $a \wedge b = \min\{a, b\}$. *This set* $\mathbb{S}_{\mathbf{p}} \subset \mathbb{S}^2$ *is nonempty.*

*For any vector* $\mathbf{q}_1 \in \mathbb{S}_{\mathbf{p}}$, *we define the following vectors*

$$\mathbf{q}_3 = (1 + \frac{p_2}{p_3})\mathbf{p} - \frac{p_2}{p_3}\mathbf{q}_1, \qquad \mathbf{q}_2 = (1 - \frac{p_3}{p_2})\mathbf{q}_1 + \frac{p_3}{p_2}\mathbf{q}_3.$$

*It is easy to see that* $\mathbf{q}_2, \mathbf{q}_3 \in \mathbb{S}^2$. *By means of stochastic vectors* $\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \mathbf{p}$, *we define the following square stochastic matrices*

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbf{p} \\ \mathbf{p} \\ \mathbf{p} \end{pmatrix}, \qquad \mathbb{Q}_2 = \begin{pmatrix} \mathbf{p} \\ \mathbf{q}_2 \\ \mathbf{q}_1 \end{pmatrix}, \qquad \mathbb{Q}_3 = \begin{pmatrix} \mathbf{p} \\ \mathbf{q}_1 \\ \mathbf{q}_3 \end{pmatrix}.$$

*Due to the construction of stochastic vectors* $\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}$, *we have that* $\mathbf{p}\mathbb{Q}_1 = \mathbf{p}\mathbb{Q}_2 = \mathbf{p}\mathbb{Q}_3 = \mathbf{p}$, *i.e.*, $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{p}]$.

*Consequently,* $\mathcal{Q}_{\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}} : \mathbb{S}^2 \to \mathbb{S}^2$ *constructed by above square stochastic matrices* $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$

$$\mathcal{Q}_{\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}}(\mathbf{x}) = x_1^2\mathbf{p} + x_2^2\mathbf{q}_2 + x_3^2\mathbf{q}_3 + 2x_1x_2\mathbf{p} + 2x_1x_3\mathbf{p} + 2x_2x_3\mathbf{q}_1$$

*is an* $r-$*majorizing QSO.*

*We may get more concrete examples by choosing* $\mathbf{p}$.

*Let* $\mathbf{p}_0 = (0.7, 0.2, 0.1)$. *Then* $s = 0.3$ *and* $t = 0.5$. *Moreover, we have that*

$$\mathbb{S}_{\mathbf{p}_0} = \big\{\mathbf{x} \in \mathbb{S}^2 : \ 0 < x_3 < 0.15, \ 0 < x_2 < 0.3, \ 0.55 < x_1 < 1\big\}.$$

*This set* $\mathbb{S}_{\mathbf{p}_0}$ *is nonempty and continuum. We pick up any vector* $\mathbf{q}_1 \in \mathbb{S}_{\mathbf{p}_0}$, *say* $\mathbf{q}_1 = (0.8, 0.15, 0.05)$. *We find vectors* $\mathbf{q}_3 = 3\mathbf{p}_0 - 2\mathbf{q}_1 = (0.5, 0.3, 0.2)$ *and* $\mathbf{q}_2 = 0.5\mathbf{q}_1 + 0.5\mathbf{q}_3 = (0.65, 0.225, 0.125)$.

*It is worth of mentioning that by choosing any vector* $\mathbf{q}_1 \in \mathbb{S}_{\mathbf{p}_0}$ *and defining* $\mathbf{q}_3 = 3\mathbf{p}_0 - 2\mathbf{q}_1$ *and* $\mathbf{q}_2 = 0.5\mathbf{q}_1 + 0.5\mathbf{q}_3$, *we may obtain a plenty of examples for the* $r-$*majorizing QSO.*

**Example 3.2.** *Let* $\mathbf{p} \in \mathbb{S}^2$ *and* $\mathbf{p} > 0$. *Without loss of generality, we may assume that* $0 < p_3 \leq p_2 \leq p_1 < 1$. *Let* $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = (1, 1, 1)$.

*Let us choose any stochastic vector* $\mathbf{r}_3 \in \mathbb{S}^2$. *Since* $p_3 \leq p_2 \leq p_1$, *we have that*

$$0 \leq p_3(\mathbf{e} - \mathbf{r}_3) = p_3\mathbf{e} - p_3\mathbf{r}_3 \leq \mathbf{p} - p_3\mathbf{r}_3 \leq \mathbf{p} < \mathbf{e}.$$

*Therefore,* $\mathbf{r} = \frac{1}{1-p_3}\mathbf{p} - \frac{p_3}{1-p_3}\mathbf{r}_3 \in \mathbb{S}^2$ *is a stochastic vector. We define stochastic vectors* $\mathbf{r}_1 = \frac{p_3}{p_1}\mathbf{r}_3 + (1 - \frac{p_3}{p_1})\mathbf{r}$, $\mathbf{r}_2 = \frac{p_3}{p_2}\mathbf{r}_3 + (1 - \frac{p_3}{p_2})\mathbf{r}$ *and square stochastic matrices*

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r} \\ \mathbf{r} \end{pmatrix}, \qquad \mathbb{Q}_2 = \begin{pmatrix} \mathbf{r} \\ \mathbf{r}_2 \\ \mathbf{r} \end{pmatrix}, \qquad \mathbb{Q}_3 = \begin{pmatrix} \mathbf{r} \\ \mathbf{r} \\ \mathbf{r}_3 \end{pmatrix}.$$

*Due to the construction of stochastic vectors* $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r}$, *the square stochastic matrices* $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ *have a common stationary distribution* $\mathbf{p}$, *i.e,* $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SM}[\mathbf{p}]$. *Consequently,* $\mathcal{Q}_{\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ *constructed by above square stochastic matrices* $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$

$$\mathcal{Q}_{\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r}}(\mathbf{x}) = x_1^2\mathbf{r}_1 + x_2^2\mathbf{r}_2 + x_3^2\mathbf{r}_3 + 2(x_1x_2 + x_1x_3 + x_2x_3)\mathbf{r}$$

*is the* $r-$*majorizing QSO.*

*This example also shows that we may get a plenty of examples for the* $r-$*majorizing QSO by choosing any stochastic vector* $\mathbf{r}_3 \in \mathbb{S}^2$.

# 4. Non-scrambling $R$-majorizing Quadratic Stochastic Operators

Recall that a square stochastic matrix $\mathbb{P}$ is called *scrambling* if for any $i, j$ there is $k$ such that $p_{ik}p_{jk} > 0$. In other words, a square stochastic matrix $\mathbb{P}$ is *scrambling* if and only if any two rows are not orthogonal. We denote the set of all scrambling stochastic matrices by $\mathsf{SSM}$.

**Definition 4.1.** *The QSO* $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ *given by* (3) *is said to be scrambling if one has that* $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \in \mathsf{SSM}$, *i.e., all square stochastic matrices* $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ *are scrambling.*

We know that any scrambling square stochastic matrix $\mathbb{P}$ is regular, i.e., for any $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$, a trajectory $\{\mathbf{x}^{(n)}\}$ of $\mathbb{P}$, where $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)}\mathbb{P}$, converges to a unique stationary distribution of $\mathbb{P}$. The similar result was proven for scrambling $r-$majorizing QSO in ref. Saburov and Yusof (2014). Namely, it was proven that the scrambling $r-$majorizing QSO has a unique fixed point and its trajectory converges to its unique fixed point.

Particularly, $\mathcal{Q}_{\mathbf{q_1},\mathbf{q_2},\mathbf{q_3}} : \mathbb{S}^2 \to \mathbb{S}^2$ given by Example 3.1 is the scrambling $r-$majorizing QSO and its trajectory converges to its unique fixed point $\mathbf{p}$. Moreover, $\mathcal{Q}_{\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ given by Example 3.2 is the scrambling $r-$majorizing QSO if and only if $(\mathbf{r}_i, \mathbf{r}) \neq 0$ for any $i = 1, 2, 3$. In this case, it is regular, i.e., any trajectory starting from any initial point converges to its unique fixed point.

In this section, we are aiming to study the dynamics of some non-scrambling $r-$majorizing QSO. All examples show that the dynamics of non-scrambling $r-$majorizing QSO is completely different from the dynamics of scrambling $r-$majorizing QSO.

Let us consider a non-scrambling $r-$majorizing QSO $\mathcal{Q}_{\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ given by Example 3.2 for the special choice of stochastic vectors $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r}$,

Let $\mathbf{p}^\circ = (p, p, 1-2p)$, $\mathbf{r} = (\frac{1}{2}, \frac{1}{2}, 0)$, $\mathbf{r}_1 = \mathbf{r}_2 = (\frac{3p-1}{2p}, \frac{3p-1}{2p}, \frac{1-2p}{p})$, and $\mathbf{r}_3 = \mathbf{e}_3 = (0, 0, 1)$ where $\frac{1}{3} < p < \frac{1}{2}$. We define the following square stochastic matrices

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r} \\ \mathbf{r} \end{pmatrix}, \qquad \mathbb{Q}_2 = \begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r} \end{pmatrix}, \qquad \mathbb{Q}_3 = \begin{pmatrix} \mathbf{r} \\ \mathbf{r} \\ \mathbf{e}_3 \end{pmatrix}.$$

and the following QSO $\mathcal{Q}_{\mathbf{r_1},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ associated with above square stochastic matrices

$$\mathcal{Q}_{\mathbf{r_1},\mathbf{r}}(\mathbf{x}) = (x_1^2 + x_2^2)\mathbf{r}_1 + x_3^2\mathbf{e}_3 + 2(x_1x_2 + x_1x_3 + x_2x_3)\mathbf{r}. \tag{11}$$

**Proposition 4.1.** *Let* $\mathcal{Q}_{\mathbf{r_1},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ *be a QSO given by* (11). *The following statements hold true:*

- *(i)* $\mathcal{Q}_{\mathbf{r_1},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ *is a non-scrambling $r-$majorizing QSO w.r.t.* $\mathbf{p}^\circ$*;*
- *(ii) One has that* $Fix(\mathcal{Q}_{\mathbf{r_1},\mathbf{r}}) = \{\mathbf{e}_3, \mathbf{p}^\circ\}$*;*
- *(iii) A trajectory* $\{\mathbf{x}^{(n)}\}$ *of* $\mathcal{Q}_{\mathbf{r_1},\mathbf{r}}$ *converges to* $\mathbf{p}^\circ$ *for any* $\mathbf{x}^{(0)} \in \mathbb{S}^2 \setminus \{\mathbf{e}_3\}$*.*

*Proof.* Let $\mathcal{Q}_{\mathbf{r_1},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ be a QSO given by (11). We then get that

$\mathcal{Q}_{\mathbf{r_1},\mathbf{r}}(\mathbf{x}) = \mathbf{x}' = (x_1', x_2', x_3')$

$$\mathcal{Q}_{\mathbf{r_1},\mathbf{r}} : \begin{cases} x_1' = \frac{3p-1}{2p}(x_1^2 + x_2^2) + x_1x_2 + x_1x_3 + x_2x_3 \\ x_2' = \frac{3p-1}{2p}(x_1^2 + x_2^2) + x_1x_2 + x_1x_3 + x_2x_3 \\ x_3' = \frac{1-2p}{p}(x_1^2 + x_2^2) + x_3^2 \end{cases}$$

It is clear that $\mathbf{p}^\circ\mathbb{Q}_1 = \mathbf{p}^\circ\mathbb{Q}_2 = \mathbf{p}^\circ\mathbb{Q}_3 = \mathbf{p}^\circ$ and $\mathbb{Q}_3$ is not scrambling matrix. Therefore, $\mathcal{Q}_{\mathbf{r_1},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ is a non-scrambling $r-$majorizing QSO w.r.t. $\mathbf{p}^\circ$.

Moreover, it is clear that $x_1' = x_2'$ for any $\mathbf{x}^{(0)} \in \mathbb{S}^2$. Therefore, we shall study the dynamics of $\mathcal{Q}_{\mathbf{r_1},\mathbf{r}} : \mathbb{S}^2 \to \mathbb{S}^2$ on the set $x_1 = x_2$. In this case, one has that $x_3' = \frac{1-2p}{2p}(1-x_3)^2 + x_3^2$. Let $f(t) = \frac{1-2p}{2p}(1-t)^2 + t^2$ be a function on [0,1]. We then have that $x_3' = f(x_3)$. One can easily check that $Fix(f) = \{1-2p, 1\}$ and its trajectory converges to the fixed point $1-2p$ whenever $\frac{1}{3} < p < \frac{1}{2}$ and $t \in [0,1)$. Consequently, we have that $Fix(\mathcal{Q}_{\mathbf{r_1},\mathbf{r}}) = \{\mathbf{e}_3, \mathbf{p}^\circ\}$ and $x_3^{(n)} \to 1-2p$. Since $x_1^{(n)} = x_2^{(n)} = \frac{1-x_3^{(n)}}{2} \to p$, the trajectory $\{\mathbf{x}^{(n)}\}$ converges to $\mathbf{p}^\circ$ for any initial point $\mathbf{x}^{(0)} \in \mathbb{S}^2 \setminus \{\mathbf{e}_3\}$. This completes the proof. □

Let us consider another non-scrambling $r-$majorizing QSO.

Let $\mathbf{p}_0 = (0, p_2, p_3) \in \mathbb{S}^2$ be a stochastic vector and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be vertices of the simplex $\mathbb{S}^2$. Let us define the following square stochastic matrices

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \qquad \mathbb{Q}_2 = \begin{pmatrix} \mathbf{e}_2 \\ \mathbf{p}_0 \\ \mathbf{p}_0 \end{pmatrix}, \qquad \mathbb{Q}_3 = \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{p}_0 \\ \mathbf{p}_0 \end{pmatrix}.$$

We define the following QSO $\mathcal{Q}_{\mathbf{p}_0} : \mathbb{S}^2 \to \mathbb{S}^2$ associated with above square stochastic matrices

$$\mathcal{Q}_{\mathbf{p}_0}(\mathbf{x}) = x_1^2\mathbf{e}_1 + (x_2^2 + x_3^2)\mathbf{p}_0 + 2x_1x_2\mathbf{e}_2 + 2x_1x_3\mathbf{e}_3 + 2x_2x_3\mathbf{p}_0. \tag{12}$$

**Proposition 4.2.** *Let $\mathcal{Q}_{\mathbf{p}_0} : \mathbb{S}^2 \to \mathbb{S}^2$ be a QSO given by (12). The following statements hold true:*

*(i)* $\mathcal{Q}_{\mathbf{p}_0} : \mathbb{S}^2 \to \mathbb{S}^2$ *is a non-scrambling $r-$majorizing QSO w.r.t.* $\mathbf{p}_0$*;*

*(ii) One has that* $Fix(\mathcal{Q}_{\mathbf{p}_0}) = \{\mathbf{e}_1, \mathbf{p}_0\}$*;*

*(iii) A trajectory* $\{\mathbf{x}^{(n)}\}$ *of* $\mathcal{Q}_{\mathbf{p}_0}$ *converges to* $\mathbf{p}_0$ *for any* $\mathbf{x}^{(0)} \in \mathbb{S}^2 \setminus \{\mathbf{e}_1\}$

*Proof.* Let $\mathcal{Q}_{\mathbf{p}_0} : \mathbb{S}^2 \to \mathbb{S}^2$ be a QSO given by (12). We then get that

$$\mathcal{Q}_{\mathbf{p}_0}(\mathbf{x}) = \left(x_1^2;\ (x_2+x_3)^2 p_2 + 2x_1x_2;\ (x_2+x_3)^2 p_3 + 2x_1x_3\right).$$

$(i)$. It is clear that $\mathbf{p}_0\mathbb{Q}_1 = \mathbf{p}_0\mathbb{Q}_2 = \mathbf{p}_0\mathbb{Q}_3 = \mathbf{p}_0$ and $\mathbb{Q}_1$ is not scrambling matrix. Therefore, $\mathcal{Q}_{\mathbf{p}_0} : \mathbb{S}^2 \to \mathbb{S}^2$ is a non-scrambling $r-$majorizing QSO w.r.t. $\mathbf{p}_0$.

$(ii)$. It is clear that $\mathcal{Q}_{\mathbf{p}_0}(\mathbf{x}) = \mathbf{x}$ if and only if $x_1 = 0$ or $x_1 = 1$. If $x_1 = 1$ then $x_2 = x_3 = 0$. If $x_1 = 0$ then $x_2 + x_3 = 1$ and $\mathbf{x} = \mathcal{Q}_{\mathbf{p}_0}(0, x_2, x_3) = \mathbf{p}_0$. Therefore, $Fix(\mathcal{Q}_{\mathbf{p}_0}) = \{\mathbf{e}_1, \mathbf{p}_0\}$.

$(iii)$. Let $\{\mathbf{x}^{(n)}\}$, where $\mathbf{x}^{(n+1)} = \mathcal{Q}_{\mathbf{p}_0}(\mathbf{x}^{(n)})$, be the trajectory of QSO given by (12) starting from $\mathbf{x}^{(0)} \in \mathbb{S}^2 \setminus \{\mathbf{e}_1\}$. We then get that $x_1^{(n+1)} = (x_1^{(n)})^2 = \left(x_1^{(0)}\right)^{2^n}$. Therefore, $x_1^{(n)}$ tends to 0. On the other hand, we have that

$$\begin{aligned}
\left|x_2^{(n+1)} - p_2\right| &= \left|p_2\left((x_1^{(n)})^2 - 2x_1^{(n)}\right) + 2x_1^{(n)}x_2^{(n)}\right| \\
&\leq 3p_2x_1^{(n)} + 2x_1^{(n)} = (2+3p_2)x_1^{(n)}, \\
\left|x_3^{(n+1)} - p_3\right| &= \left|p_3\left((x_1^{(n)})^2 - 2x_1^{(n)}\right) + 2x_1^{(n)}x_3^{(n)}\right| \\
&\leq 3p_3x_1^{(n)} + 2x_1^{(n)} = (2+3p_3)x_1^{(n)}.
\end{aligned}$$

Consequently, we get that $x_2^{(n)}, x_3^{(n)} \to 0$. This means that the trajectory $\{\mathbf{x}^{(n)}\}$ converges to $\mathbf{p_0}$ for any initial point $\mathbf{x}^{(0)} \in \mathbb{S}^2 \setminus \{\mathbf{e}_1\}$. This completes the proof. □

**Remark 4.1.** *We can see that, unlike the scrambling $r-$majorizing QSO, the non-scrambling $r-$majorizing QSO may have many (more than one) fixed points.*

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# References

Lyubich, Y. I. *Mathematical Structures in Population Genetics.* Springer-Verlag, Berlin, 1992.

Ganikhodjaev, N., Saburov, M., & Uygun, J. Mendelian and non-mendelian quadratic operators. *App. Math. Info. Sci.*, 7(5):1721–1729, 2013.

Ganikhodzhaev, R., Mukhamedov, F., & Rozikov, U. Quadratic stochastic operators and processes: Results and open problems. *Inf. Dim. Anal. Quantum Prob. Rel. Top. (IDAQPRT)*, 14(2):279–335, 2011.

Mukhamedov, F. & Saburov, M. On homotopy of volterrian quadratic stochastic operators. *Appl. Math. Info. Sci.*, 4(1):47–62, 2010.

Mukhamedov, F. & Saburov, M. On dynamics of lotka-volterra type operators. *Bull. Malays. Math. Sci. Soc.*, 37(1):59–64, 2014.

Mukhamedov, F., Saburov, M., & Qaralleh, I. On $\xi^{(s)}-$quadratic stochastic operators on two dimensional simplex and their behavior. *Abs. App. Anal.*, 2013:12, 2013.

Saburov, M. Some strange properties of quadratic stochastic volterra operators. *World Applied Science Journal*, 21:94–97, 2013.

Joe, H. Majorization and divergence. *J. Math. Anal. App.*, (148):287–305, 1990.

Marshall, A. W., Olkin, I., & Arnold, B. C. *Inequalities: Theory of majorization and its applications.* Springer, 2 edition, 2011.

Ganikhodzhaev, R. On the definition of quadratic doubly stochastic operators. *Russian Math. Surveys*, 48:244–246, 1992.

Saburov, M. & Saburov, K. Mathematical models of nonlinear uniform consensus. (40):306–312, 2014a.

Saburov, M. & Saburov, K. Reaching a nonlinear consensus: Polynomial stochastic operators. 12(6):1276–1282, 2014b.

Saburov, M. & Yusof, N. A. Scrambling $r-$majorizing quadratic stochastic operators. Submitted to Ukrainian Mathematical Journal, 2014.